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PART III. THE DISPERSION, UNDER GRAVITY, OF A COLUMN OF FLUID SUPPORTED ON A RIGID HORIZONTAL PLANE*

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CONTENTS

The dispersive motion of a fluid column, collapsing under its own weight whilst immersed in a lighter fluid, both fluids resting on a rigid horizontal bottom, is idealized as a symmetric problem of incompressible potential flow. The velocity potentials in the two fluids are functions of two space-variables and time, and satisfy Laplace's equation in the two space-variables; and the principal boundary conditions are that, at every point of the fluid interface, the two fluid pressures are equal, and the two normal fluid velocities are both equal to the normal velocity of the interface itself. The initial accelerations of the boundary of a column of semicircular cross-section are derived analytically.

An approximate numerical method of solution for the early stages of such a motion is obtained by satisfying these boundary conditions at only a finite number of angular positions instead of everywhere on the fluid interface. Some calculations by this method are shown for certain fluid columns *in vacuo*.

Alternatively, by neglecting vertical accelerations in the motion, the problem is reduced to one of hyperbolic type in one space-variable and time, and this approximation may be solved by the numerical method of characteristics. Some calculations of this type are also shown, in which vertical accelerations have been neglected *ab initio*, and which are therefore appropriate to initially squat columns.

The hydrodynamical problem of the collapse of a fluid column surrounded by a second lighter fluid, both resting on a rigid horizontal plane, was suggested by the 'base surge' observed at the Atomic Weapon Trials at Bikini.

1. Introduction

An interesting hydrodynamical problem is that of the dispersion under gravity of a fluid column immersed in a surrounding fluid medium of lower density and resting on a rigid horizontal plane. Examples of such a dispersive motion are afforded by the bursting of a dam wall, the sudden shattering of a vessel containing liquid, or the spread of a thin pancake mixture in a frying pan.

Another striking example, and the one which inspired the investigation described in this paper, arises from the fact that a large explosion in shallow water throws a quantity of water vertically upwards and forms a column consisting of a mixture of fine water droplets and air. Soon after the explosion, this column starts to collapse under gravitational forces and, at the same time, spreads out rapidly from the base, producing the phenomenon of the 'base surge' which was observed in the atomic bomb test at Bikini Lagoon. The Bikini

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base surge has been described and illustrated in *The effects of atomic weapons* (United States Atomic Energy Commission 1950) and in *Bombs at Bikini* (W. A. Shurcliff 1947). A photograph of the base surge at Bikini is reproduced in part V by Martin & Moyce. In the case of an atomic weapon, the base surge accompanying the collapse of the fluid column has great practical importance, since it is thought to contain most of the deadly fission products.

If the water droplets are sufficiently fine, the early motion of collapse will be similar to that of a column of a single fluid slightly denser than air, and, to this extent, the results of the ideal mathematical treatment described in this paper may be applied to the early motion of the base surge. Ultimately the water drops will begin to sediment downwards through the air, and will acquire in this way a differential vertical motion through the air.

The problem of the collapsing liquid column in a second liquid may be idealized into one of incompressible potential flow, neglecting viscosity, ground friction and turbulence, and is easily shown to be of parabolic type in three independent variables (two space-variables, since the column has plane or axial symmetry, and time); for the velocity potentials, which are functions of all three variables, satisfy Laplace's equation in the two space-variables.

The principal boundary conditions are that, at every point of the unknown moving boundary of the collapsing fluid column, the two fluid pressures must be equal, and the two normal fluid velocities must both be equal to the normal velocity of the boundary. The boundary surface is therefore a slip-stream, although of course, when viscous terms are included, there will be a boundary layer in both fluids. The initial accelerations of the boundary of a column of semicircular cross-section can be derived analytically, as shown in § 3.

The solution may be developed, in general, in terms of infinite series of cosines (plane symmetry) or spherical harmonics (axial symmetry), in which the coefficients are functions of time. A numerical solution for the early stages of the motion may be obtained by limiting the above series to a finite number of terms. The boundary conditions can then only be satisfied at a finite number of angular positions, instead of everywhere, on the column boundary; but, in this way, the solution may be reduced numerically to that of a finite number of simultaneous linear equations with constant coefficients at each small interval of time. The fewer terms retained in the series and thus the fewer points at which the boundary conditions are satisfied, the shorter is the duration in time before the numerical solution ceases to satisfy reasonably the principles of conservation of mass and energy.

The numerical solutions given are for fluid columns of initially semicircular cross-section *in vacuo*, for which the number of simultaneous equations to be solved is much reduced. In all cases the column spreads out rapidly from the base, as would be expected.

An approximate solution, appropriate to the later stages of such a dispersive motion, or to columns which are initially very squat, may be obtained by neglecting vertical accelerations in the fluid motion. In this case the problem reduces to one of hyperbolic type in two independent variables (one space-variable and time), and a solution is readily obtained by the numerical method of characteristics. It is shown that, with this approximation, the dispersive motions of all columns which are similar, except for a scale factor in height, are derivable from the same characteristic solution, and that differences in the relative densities of the column and the surrounding medium correspond to different values of gravity.

Numerical examples are given in which vertical accelerations have been neglected *ab initio*, and which are therefore appropriate to initially squat columns.

The method of characteristics may be used to continue solutions obtained by other methods for the collapse of columns which were not originally squat.

2. The general problem

The motion is assumed to start from rest, and is taken as symmetrical about either a vertical plane or a vertical axis, so that, in either case, polar co-ordinates (r, θ) may be used, with the origin at the centre of the base of the column and $\theta = 0$ vertical. The motion is dependent on the three variables r, θ and the time t; and partial derivatives of functions with respect to r, θ or t will be denoted by $\partial/\partial r$, $\partial/\partial\theta$ and $\partial/\partial t$. The external boundary of the fluid column when there is only the one fluid, or the interface between the two fluids where there are two, may be written $r = R(\theta, t)$. Partial derivatives with respect to θ or t at points on the interface will be denoted by $\delta/\delta\theta$, $\delta/\delta t$. Thus $\delta f_1/\delta\theta$ denotes the rate of change with respect to θ along the boundary surface, of the function f_1 in the medium 1 at time t. The suffixes 1, 2 will be used to refer to the denser fluid within the column and the lighter surrounding fluid respectively (see figure 1).

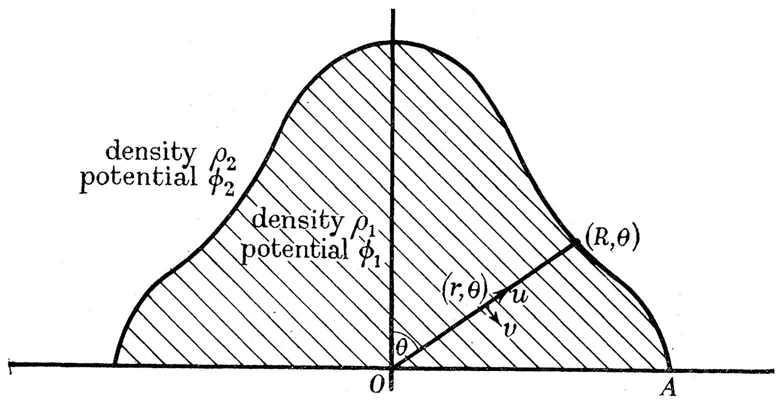

Figure 1. Collapsing column of fluid 1 surrounded by fluid 2, both resting on rigid horizontal bottom. The figure shows the notation used in the text.

The effects of compressibility, viscosity, etc., will be neglected. Then, since the motion starts from rest, and is due only to gravitational forces, it may be described in each fluid by a potential function $\phi = \phi(r, \theta, t)$ defined so that

$$u = -\frac{\partial\phi}{\partial r}, \quad v = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

are, respectively, the radial and transverse components of fluid velocity at the point (r, θ) at the time t.

The equations of motion integrate to give

$$\frac{p}{\rho} = \frac{\partial\phi}{\partial t} - \tfrac{1}{2}q^2 - gr\cos\theta, \tag{1}$$

where p and ρ are respectively the pressure and density of the relevant fluid, and $q = (u^2+v^2)^{\frac{1}{2}}$ is the fluid velocity.

The equation of conservation of mass reduces to

$$\nabla^2_{r,\theta}(\phi) = 0, \tag{2}$$

i.e. Laplace's equation, except that ϕ is not a function of r and θ only, but depends also on t.

At the fluid interface, two boundary conditions must be satisfied. First, the velocity of each fluid normal to the interface must be equal to the normal velocity of the interface itself. After some reduction, this gives the relations

$$\frac{1}{R^2}\frac{\delta R}{\delta \theta}\left[\frac{\partial \phi_1}{\partial \theta}\right]_R - \left[\frac{\partial \phi_1}{\partial r}\right]_R = \frac{\delta R}{\delta t} = \frac{1}{R^2}\frac{\delta R}{\delta \theta}\left[\frac{\partial \phi_2}{\partial \theta}\right]_R - \left[\frac{\partial \phi_2}{\partial r}\right]_R, \tag{3}$$

where, for example, $[\partial\phi_1/\partial\theta]_R$ denotes the value of $\partial\phi_1/\partial\theta$ on the interface $r = R(\theta, t)$.

Secondly, the pressures in the two fluids must be equal at the interface, and thus

$$\rho_1\left[\frac{\partial \phi_1}{\partial t}\right]_R - \tfrac{1}{2}\rho_1[q_1^2]_R - g\rho_1 R\cos\theta = \rho_2\left[\frac{\partial \phi_2}{\partial t}\right]_R - \tfrac{1}{2}\rho_2[q_2^2]_R - g\rho_2 R\cos\theta. \tag{4}$$

On the horizontal base-plane, $\theta = \frac{1}{2}\pi$, the motion is entirely horizontal; and similarly, on the vertical plane or axis of symmetry, $\theta = 0$, the motion is entirely vertical; hence the conditions

$$\theta = 0 \text{ or } \tfrac{1}{2}\pi: \quad \frac{\partial \phi_1}{\partial \theta} = \frac{\partial \phi_2}{\partial \theta} = 0. \tag{5}$$

The conditions of zero velocity at the origin, and no disturbance at infinity, give

$$\left.\begin{aligned} r = 0: &\quad \frac{\partial \phi_1}{\partial r} = 0, \quad \frac{\partial \phi_1}{\partial \theta} = 0, \\ r \to \infty: &\quad \frac{\partial \phi_2}{\partial r} = 0, \quad \frac{\partial \phi_2}{\partial \theta} = 0. \end{aligned}\right\} \tag{6}$$

Finally, there is an initial condition defining the original fluid interface,

$$t = 0: \quad r = R(\theta, 0), \tag{7}$$

in addition to the initial condition of rest already assumed.

The solutions of the potential equation (2) which satisfy the conditions (5) and (6) may be written, in plane symmetry,

$$\left.\begin{aligned} \phi_1 &= (ga^3)^{\frac{1}{2}} \sum_{n=0}^{\infty} A_{2n}(r/a)^{2n}\cos 2n\theta, \\ \phi_2 &= (ga^3)^{\frac{1}{2}} \sum_{n=0}^{\infty} B_{2n}(r/a)^{-2n}\cos 2n\theta, \end{aligned}\right\} \tag{8}$$

and in axial symmetry,

$$\left.\begin{aligned} \phi_1 &= (ga^3)^{\frac{1}{2}} \sum_{n=0}^{\infty} A_{2n}(r/a)^{2n} P_{2n}(\cos\theta), \\ \phi_2 &= (ga^3)^{\frac{1}{2}} \sum_{n=0}^{\infty} B_{2n}(r/a)^{-2n-1} P_{2n}(\cos\theta), \end{aligned}\right\} \tag{9}$$

where the coefficients A_{2n}, B_{2n} are non-dimensional functions of t only and a is some representative length, say the initial radius of the base of the column.

We come now to consider the form of the Fourier expansion of the radius vector $R(\theta, t)$ in terms of θ. One naturally uses the same set of orthogonal functions as was used in the expansion of the velocity potentials, and expands $R(\theta, t)$ in the range $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ as follows:

$$\left.\begin{aligned} \frac{R}{a} &= \sum_{n=0}^{\infty} S_{2n} \cos 2n\theta, \\ \frac{R}{a} &= \sum_{n=0}^{\infty} S_{2n} P_{2n}(\cos\theta), \end{aligned}\right\} \tag{10}$$

for the two cases of plane and axial symmetry respectively. The coefficients S_{2n} are non-dimensional functions of t only.

The normal procedure would be to substitute (8) [or (9)] and (10) in (3) and (4) and attempt to manipulate all the trigonometric functions so that the only trigonometric functions finally appearing are those of the orthogonal sets which are being used as the bases, namely, $\cos 2n\theta$ or $P_{2n}(\cos\theta)$. By collecting together all the terms in each particular $\cos 2n\theta$ or $P_{2n}(\cos\theta)$, the equations (3) and (4) yield three infinities of equations. These equations would, in principle, be solved successively, beginning with the gravest harmonic and moving to each higher harmonic in turn. If such a procedure were in fact possible, we would hope to have established a convergent process, and to obtain a reasonable approximation to the motion with the first few terms. Unfortunately, however, this procedure is too complicated to give satisfactory results. The high powers of R which appear in the velocity potentials give rise to very high order equations in the S coefficients, when the expansions (8) or (9) are used in (3) and (4). An alternative procedure was therefore developed.

In numerical work, of course, one can only take a few terms in the expansion. Any *finite* number of terms in the expansion (10) imposes the result that

$$\left[\frac{\delta R}{\delta\theta}\right]_{\theta=\frac{1}{2}\pi} = 0, \tag{11}$$

and therefore that the interface of the two fluids is always perpendicular at the ground. Later in the paper we shall give a demonstration that the interface at large times meets the ground at an angle $\frac{1}{3}\pi$. Presumably, therefore, as soon as the motion starts (i.e. $t>0$), the interface at the ground must depart from the vertical. Indeed, if we wish, we can start with a liquid column which does not meet the ground vertically. The expansions (10) could still be made, but would be formally acceptable only in the limit of infinite expansions. Any approximate numerical solution, using a finite number of harmonics in (10), would involve the approximation that the angle of contact at the ground is $\frac{1}{2}\pi$, but one would expect that the representation found for the motion, apart from this blemish, would improve with the number of terms taken in the expansion.

We shall later describe how we have solved (3) and (4) by approximate numerical methods, using a finite number of terms of the expansions (8), (9) and (10). Relying on the intuitive hypothesis that the lower harmonics in ϕ are the most important, we have, for this purpose, used the actual expansions (10), stopping at $n = 3$ or $n = 4$, thus making the angle of contact with the ground always equal to $\frac{1}{2}\pi$.

It was later realized that, in so far as our methods do not use the orthogonality properties of the harmonics, we could have used expansions such as

$$\left.\begin{aligned} \frac{R}{a} &= \sum_{n=0}^{l} T_n \cos n\theta, \\ \frac{R}{a} &= \sum_{n=0}^{l} T_n P_n(\cos\theta), \end{aligned}\right\} \tag{10a}$$

or any other convenient form of expansion, such as a polynomial in θ, which would not impose the condition of an angle of contact $\frac{1}{2}\pi$ with the ground. Calculations based on such finite expansions are now proceeding, and if the results are of additional interest they will be published in a later paper.

We have actually made three separate attacks on solving the motion. The first deals only with the initial accelerations. It was found possible in this case to retain infinite expansions for ϕ and R, and to get the complete solution, including the singularity at $\theta = \frac{1}{2}\pi$. Details follow later (§ 3). The third method uses rectangular co-ordinates and equates the vertical acceleration everywhere to zero. In this case, a complete solution is also obtained. This solution, of course, can only be applied to the motion of columns whose vertical height is small compared with their horizontal dimensions. Details are given later (§ 5). The second method of solution is perhaps the most interesting and will be outlined now, although the mathematical details are given later (§ 4).

The essential problem is to solve by some arithmetical method of approximation the two equations (3) and (4). We have already reached the conclusion that a complete solution is impossible, and our more limited objective now is to approximate to ϕ with the few lowest harmonics. Because we have only a few harmonics, equations (3) and (4) cannot be satisfied exactly. We therefore argued as follows. Let us attempt to satisfy these two equations at a limited number of values of θ, rather than over the whole range, i.e. divide up the range $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ into $(N-1)$ parts, and attempt to satisfy (3) and (4) at

$$\theta_0 = 0, \quad \theta_1 = \tfrac{1}{2}\pi/(N-1), \quad \ldots, \quad \theta_{N-1} = \tfrac{1}{2}\pi.$$

Thus, the pressure will be continuous from one fluid to the other only at N values of θ, and the boundary surfaces of the two fluids will intersect only at these same N values. In the special case when there is no second fluid surrounding the column, the pressure will be zero at the boundary only for these N values of θ, and an artificial pressure distribution is thus introduced over the 'free' surface, having nodes at these N angular values. We regard the unknowns as

$$R_0, R_2, \ldots, R_{2N-2}; \quad A_0, A_2, \ldots, A_{2N-2}; \quad B_0, B_2, \ldots, B_{2N-2}.$$

The number of unknowns is therefore $3N$ and we have exactly $3N$ equations from (3) and (4) by particularizing the values of θ to $\theta_0, \ldots, \theta_{N-1}$. At each stage of the motion, we have calculated the N values $R_0, R_2, \ldots, R_{2N-2}$. To these values we fit a curve in even harmonics of the form of the first N terms of the expansion (10). Knowing all the values at one time, we proceed to the next step in time, using in our equations the particular values of $[\delta R/\delta\theta]$ obtained from the curve fitting to R.

We have not, in fact, so far attempted a two-fluid problem, but we have carried through the process described above for the case of a single fluid *in vacuo*. The number of simultaneous equations to be solved in each step is $2N$ for the case of a single fluid, and we have used $N = 4$ and $N = 5$ in special examples.

Non-dimensional variables. In order to simplify the algebra, we use non-dimensional variables. The first group of problems we shall consider relate to the collapse of fluid columns initially of hemi-cylindrical shape, resting on the diametral section; or of hemispherical shape resting on the diametral section. Let a be the initial radius in either case. Then we introduce the non-dimensional variables

$$\left.\begin{aligned} \Phi &= \phi/(ga^3)^{\frac{1}{2}}, \\ \tau &= t(g/a)^{\frac{1}{2}}, \\ \xi &= r/a, \\ \mathscr{R} &= R/a, \end{aligned}\right\} \tag{12}$$

and we write
$$\epsilon = (\rho_1/\rho_2) - 1. \tag{13}$$

These variables are used in the following two sections.

3. The initial accelerations

This section is concerned with the initial accelerations throughout the two fluids for columns initially of hemi-cylindrical or hemispherical shape, released from rest.

In virtue of the initial rest condition, we may write for small values of τ

$$\left.\begin{aligned} A_0 &= a_{0,0} + a_{0,1}\tau + a_{0,2}\tau^2/2! + \ldots, \\ B_0 &= b_{0,0} + b_{0,1}\tau + b_{0,2}\tau^2/2! + \ldots, \\ A_{2n} &= a_{2n,1}\tau + a_{2n,2}\tau^2/2! + \ldots \quad (n > 0), \\ B_{2n} &= b_{2n,1}\tau + b_{2n,2}\tau^2/2! + \ldots \quad (n > 0), \end{aligned}\right\} \tag{14}$$

whilst, by virtue of the initial condition $\tau = 0$, $\mathscr{R} = 1$, and the initial rest condition, we may write

$$\left.\begin{aligned} S_0 &= 1, \\ S_{2n} &= s_{2n,2}\tau^2/2! + s_{2n,3}\tau^3/3! + \ldots \quad (n > 0). \end{aligned}\right\} \tag{15}$$

Substituting these values in the relations (3), differentiating with respect to τ, and setting $\tau = 0$, we obtain, in the case of plane symmetry,

$$-2na_{2n,1} = s_{2n,2} = 2nb_{2n,1} \quad (n > 0), \tag{16}$$

whilst substituting in the relation (4) gives

$$\sum_{n=0}^{\infty} (\rho_1 a_{2n,1} - \rho_2 b_{2n,1}) \cos 2n\theta = (\rho_1 - \rho_2) \cos\theta. \tag{17}$$

Using (13) to eliminate the densities, we get

$$(1+\epsilon)\, a_{0,1} - b_{0,1} + (2+\epsilon) \sum_{n=1}^{\infty} a_{2n,1} \cos 2n\theta = \epsilon \cos\theta. \tag{18}$$

Expand $\cos\theta$ in the range $(0, \frac{1}{2}\pi)$, as a series in $\cos 2n\theta$, to give

$$\cos\theta = \frac{2}{\pi}\left[1 + \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{(4n^2-1)} \cos 2n\theta\right]. \tag{19}$$

Then, for $n>0$,
$$a_{2n,1} = (-1)^{n-1}\frac{4}{\pi}\left(\frac{\epsilon}{2+\epsilon}\right)\frac{1}{(4n^2-1)}, \tag{20}$$
$$s_{2n,2} = (-1)^{n}\frac{4}{\pi}\left(\frac{\epsilon}{2+\epsilon}\right)\frac{2n}{(4n^2-1)}, \tag{21}$$
and the initial radial acceleration of the interface is
$$\frac{4g}{\pi}\left(\frac{\epsilon}{2+\epsilon}\right)\sum_{n=1}^{\infty}(-1)^n\frac{2n}{(4n^2-1)}\cos 2n\theta. \tag{22}$$
The series (22) is convergent for $0\leqslant\theta<\frac{1}{2}\pi$, and has the sum
$$-\frac{2g}{\pi}\left(\frac{\epsilon}{2+\epsilon}\right)[1+\sin\theta\log_e\tan(\tfrac{1}{4}\pi-\tfrac{1}{2}\theta)].$$
The series is divergent at $\theta=\frac{1}{2}\pi$.

Thus, for sufficiently small values of τ, the equation of the interface in dimensional form is approximately
$$R = a-\frac{gt^2}{\pi}\left(\frac{\epsilon}{2+\epsilon}\right)[1+\sin\theta\log_e\tan(\tfrac{1}{4}\pi-\tfrac{1}{2}\theta)]. \tag{23}$$
In the particular case of a fluid column *in vacuo*, $\epsilon\to\infty$ and (23) reduces to
$$R = a-\frac{gt^2}{\pi}[1+\sin\theta\log_e\tan(\tfrac{1}{4}\pi-\tfrac{1}{2}\theta)]. \tag{23a}$$

The infinite initial acceleration at $\theta=\frac{1}{2}\pi$ implies that the foot of the column shoots out horizontally, but we are unable to say whether a jet is formed. Our equations fail at $\theta=\frac{1}{2}\pi$ because we have assumed that the displacement is small, and consequently that the normal velocity of the interface is the same as the radial velocity of the fluid. This condition holds everywhere except at $\theta=\frac{1}{2}\pi$.

In the case of axial symmetry, the relations (16) and (18) become, respectively,
$$-2na_{2n,1} = s_{2n,2} = (2n+1)\,b_{2n,1}\quad(n>0), \tag{16'}$$
$$(1+\epsilon)\,a_{0,1}-b_{0,1}+\sum_{n=1}^{\infty}\left\{\frac{(4n+1)+(2n+1)\,\epsilon}{2n+1}\right\}a_{2n,1}P_{2n}(\cos\theta) = \epsilon\cos\theta. \tag{18'}$$
Now $\cos\theta$ may be expanded in the range $(0,\frac{1}{2}\pi)$, as a series in $P_{2n}(\cos\theta)$, to give
$$\cos\theta = \frac{1}{2}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(4n+1)\,(2n-3)!}{2^{2n-1}(n+1)!\,(n-2)!}P_{2n}(\cos\theta). \tag{19'}$$
Then, $n>0$,
$$a_{2n,1} = (-1)^{n-1}\left\{\frac{(2n+1)\,(4n+1)\,\epsilon}{(4n+1)+(2n+1)\,\epsilon}\right\}\frac{(2n-3)!}{2^{2n-1}(n+1)!\,(n-2)!}, \tag{20'}$$
$$s_{2n,2} = (-1)^{n}\left\{\frac{(2n+1)\,(4n+1)\,\epsilon}{(4n+1)+(2n+1)\,\epsilon}\right\}\frac{2n!}{2^{2n}(2n-1)\,(n+1)!\,(n-1)!}, \tag{21'}$$
and, for sufficiently small values of τ, the equation of the interface in dimensional form is approximately
$$R = a-\tfrac{1}{2}gt^2\sum_{n=1}^{\infty}(-1)^n\left\{\frac{(2n+1)\,(4n+1)\,\epsilon}{(4n+1)+(2n+1)\,\epsilon}\right\}\frac{(2n)!}{2^{2n}(2n-1)\,(n+1)!\,(n-1)!}P_{2n}(\cos\theta),$$
or
$$R = a-\tfrac{1}{2}gt^2\Sigma(\epsilon,\theta). \tag{23'}$$

For small values of ϵ, that is, when the fluid column is only slightly denser than the surrounding medium,

$$\Sigma(\epsilon,\theta) = \epsilon\sum_{n=1}^{\infty}(-1)^n\frac{(2n+1)!}{2^{2n}(2n-1)\,(n+1)!\,(n-1)!}P_{2n}(\cos\theta)$$
$$= \epsilon\Sigma_0(\theta)\quad\text{(say)}\qquad(24)$$

approximately; and, for a fluid column *in vacuo*, or in a much less dense surrounding medium, $\epsilon\to\infty$ and

$$\Sigma(\epsilon,\theta) = \sum_{n=1}^{\infty}(-1)^n\frac{(4n+1)\,(2n)!}{2^{2n}(2n-1)\,(n+1)!\,(n-1)!}P_{2n}(\cos\theta)$$
$$= \Sigma_\infty(\theta)\quad\text{(say)}.\qquad(25)$$

The series $\Sigma_0(\theta)$ and $\Sigma_\infty(\theta)$ are again found to be convergent for $0\leqslant\theta<\frac{1}{2}\pi$ but divergent for $\theta=\frac{1}{2}\pi$, and have been summed by the method indicated below.

Laplace's first integral for $P_n(\cos\theta)$ gives (see, for example, Whittaker & Watson 1927, chap. XV),

$$P_n(\cos\theta) = \frac{i}{\pi}\int_{e^{-i\theta}}^{e^{i\theta}} h^n(1-2h\cos\theta+h^2)^{-\frac{1}{2}}\,dh.$$

Hence, if suitable conditions are satisfied,

$$\Sigma_0(\theta) = \frac{i}{\pi}\int_{e^{-i\theta}}^{e^{i\theta}}\Sigma_0'(h^2)\,(1-2h\cos\theta+h^2)^{-\frac{1}{2}}\,dh$$

and

$$\Sigma_\infty(\theta) = \frac{i}{\pi}\int_{e^{-i\theta}}^{e^{i\theta}}\Sigma_\infty'(h^2)\,(1-2h\cos\theta+h^2)^{-\frac{1}{2}}\,dh,$$

where $\Sigma_0'(h^2)$, $\Sigma_\infty'(h^2)$ are the corresponding power series in h^2. It is not difficult to show that, for $|h|<1$,

$$\left.\begin{aligned}\Sigma_0'(h^2) &= \frac{2}{3h^2}-\frac{2h^2+1+2/h^2}{3(1+h^2)^{\frac{1}{2}}}\\ \text{and}\quad \Sigma_\infty'(h^2) &= \frac{2}{h^2}-\frac{h^2+1+2/h^2}{(1+h^2)^{\frac{1}{2}}},\end{aligned}\right\}\qquad(26)$$

and then ultimately

$$\left.\begin{aligned}\Sigma_0(\theta) &= \tfrac{2}{3}\cos\theta-\frac{8I_1}{3\pi}+\frac{I_2}{\pi},\\ \Sigma_\infty(\theta) &= 2\cos\theta-\frac{6I_1}{\pi}+\frac{2I_2}{\pi},\end{aligned}\right\}\qquad(27)$$

where

$$\left.\begin{aligned}I_1 &= \int_0^\theta\frac{\cos^2\phi\,d\phi}{[\cos\phi(\cos\phi-\cos\theta)]^{\frac{1}{2}}},\\ I_2 &= \int_0^\theta\frac{d\phi}{[\cos\phi(\cos\phi-\cos\theta)]^{\frac{1}{2}}}.\end{aligned}\right\}\qquad(28)$$

The integral I_1 may be shown to involve elliptic integrals of the third kind, and has been computed numerically for the present purpose. The integral I_2 is an elliptic integral of the first kind and is tabulated. As $\theta\to 0$, I_1 and I_2 both tend to $\pi/\sqrt{2}$.

Table 1 gives the calculated initial radial accelerations both for the plane and axially symmetric cases, and for small and large values of ϵ. In figure 2, curve 1 shows the boundary of the plane symmetric column *in vacuo* as given by equation (23 *a*) at $t = 0{\cdot}5605(a/g)^{\frac{1}{2}}$, when the height has fallen by 10 %; and curve 2 shows the boundary of the axially symmetric column *in vacuo* as given by equation (23′) with $\epsilon \to \infty$, at $t = 0{\cdot}4627(a/g)^{\frac{1}{2}}$, when again the height has fallen by 10 %.

Table 1. Initial radial accelerations in units of g

	plane symmetry small ϵ	plane symmetry $\epsilon \to \infty$	axial symmetry small ϵ	axial symmetry $\epsilon \to \infty$
0°	$-0{\cdot}318\epsilon$	$-0{\cdot}637$	$-0{\cdot}512\epsilon$	$-0{\cdot}828$
30°	$-0{\cdot}231\epsilon$	$-0{\cdot}462$	$-0{\cdot}405\epsilon$	$-0{\cdot}665$
60°	$+0{\cdot}045\epsilon$	$+0{\cdot}089$	$-0{\cdot}084\epsilon$	$-0{\cdot}164$
85°	$+0{\cdot}675\epsilon$	$+1{\cdot}349$	$+0{\cdot}583\epsilon$	$+0{\cdot}995$
87°	$+0{\cdot}840\epsilon$	$+1{\cdot}679$	$+0{\cdot}749\epsilon$	$+1{\cdot}312$
88°	$+0{\cdot}969\epsilon$	$+1{\cdot}939$	$+0{\cdot}880\epsilon$	$+1{\cdot}563$

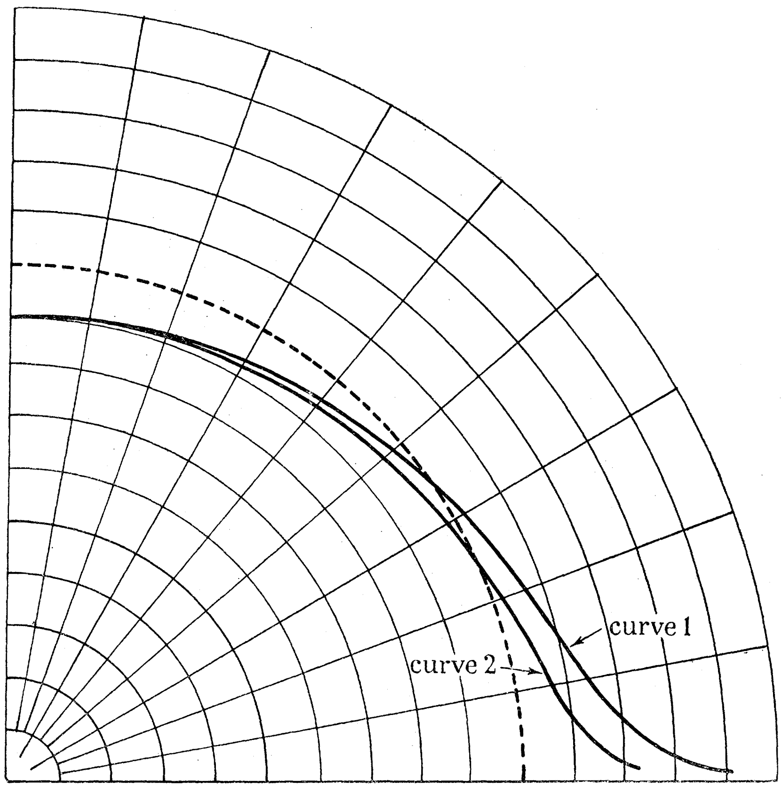

Figure 2. The shape of the plane symmetric column *in vacuo* when the height has fallen by 10 % (curve 1), and the shape of the axially symmetric column *in vacuo* when the height has fallen by 10 % (curve 2). The curves have been drawn assuming that the acceleration of each point on the surface has remained at its initial value.

4. Approximate numerical solution for the initial motion

The general solution has been reduced in § 2 to that of three ordinary first-order differential equations which must be satisfied by the functions $A_{2n}(t)$, $B_{2n}(t)$, $S_{2n}(t)$ at all radial positions from $\theta = 0$ to $\theta = \frac{1}{2}\pi$. These three equations may be considered as the limit as $N \to \infty$ of the corresponding finite system of $3N$ equations in which the series (8), (9) and (10) terminate at $n = N-1$, and the boundary conditions (3) and (4) are satisfied only for N arbitrary values of the angular variable θ instead of for all values from zero to $\frac{1}{2}\pi$.

We have already explained in § 2 that if the radius vector to the interface R is expanded in terms of only the even harmonics, then for any finite expansion, the interface must meet the ground vertically. On the other hand, it is not necessary in the numerical work to expand R only in even harmonics, and then the interface will not in general be vertical at the point where it meets the ground. Arguments can in fact be advanced to show that the interface must ultimately make an angle of $\frac{1}{3}\pi$ with the ground. We are indebted to Professor Harold Jeffreys for demonstrating this result and for permission to quote his argument. A similar proof has already been given independently by von Karman (1940) in connexion with 'gravity currents'. Both Jeffreys's and Karman's proofs are reminiscent of Stokes's proof (1880) that the crest of the highest progressive finite plane gravity wave of permanent form on deep water is a cusp of semi-angle $\frac{1}{3}\pi$. Professor Jeffreys's proof is as follows.

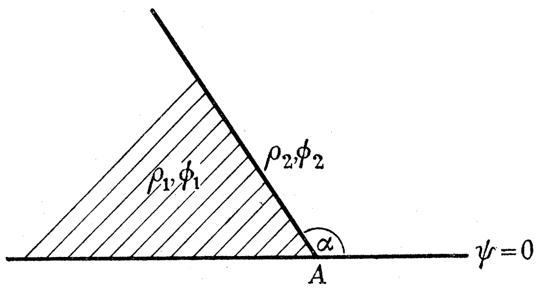

FIGURE 3. Steady-state motion at the base of the column. The point at which the column touches the bottom is kept at rest by a mass motion of the whole system.

Superpose a uniform velocity on the whole system in order to bring one point of contact between the fluid interface and the base-plane to rest, and assume that steady conditions exist near this point. If the angle of contact α (figure 3) is independent of t, the fluid interface is then a stream surface, and there are two irrotational motions in angles α, $(\pi-\alpha)$. If polar co-ordinates (r, ψ) $(\psi = 0$ horizontal) are taken about this point of contact as origin, and

$$\phi_1 = Ar^{m_1} \cos m_1 \psi + \text{terms containing higher powers of } r,$$
$$\phi_2 = Br^{m_2} \cos m_2 \psi + \text{terms containing higher powers of } r,$$

near the origin, then q is of order r^{m_1-1} or r^{m_2-1} respectively. From equation (4)

$$\rho_1(\tfrac{1}{2}q_1^2 + gr \sin \psi) = \rho_2(\tfrac{1}{2}q_2^2 + gr \sin \psi) \tag{29}$$

along the fluid interface $\psi = \alpha$, and the other boundary conditions are

$$\psi = 0: \quad \frac{\partial \phi_2}{\partial \psi} = 0,$$

$$\psi = \alpha: \quad \frac{\partial \phi_1}{\partial \psi} = \frac{\partial \phi_2}{\partial \psi} = 0,$$

$$\psi = \pi: \quad \frac{\partial \phi_1}{\partial \psi} = 0.$$

It follows that, if the densities are different, the only solution compatible with these conditions, in which the lowest powers of r in (29) can balance, is given (apart from the cases $\alpha = 0, \pi$) by

$$2(m_2-1) = 1 \quad (m_1 > m_2).$$

This leads to $m_1 = 3$, $m_2 = \frac{3}{2}$ and $\alpha = 120°$; and since both $\partial\phi_1/\partial t$ and $\partial\phi_2/\partial t$ are then of higher order than unity in r, steady motion as a local approximation appears to be justified.

Substituting $\phi_1 = Ar^3 \cos 3\psi$, $\phi_2 = Br^{\frac{3}{2}} \cos(\frac{3}{2}\psi)$ in (29) gives

$$B^2 = 4g(\rho_1 - \rho_2)/3\sqrt{3}\,\rho_2,$$

and thus B is only real if $\rho_1 > \rho_2$. It follows that there is just one possible solution, in which the denser fluid is contained within the 60° angle of contact.

We give now an example of our numerical method of solution of equations (3) and (4) in the case of axial symmetry. In this case, using the non-dimensional variables defined by (12), we have that

$$\left.\begin{aligned}\Phi_1 &= A_i \xi^i P_i(\cos\theta),\\ \Phi_2 &= B_i \xi^{-i-1} P_i(\cos\theta),\\ \mathscr{R} &= S_i P_i(\cos\theta),\end{aligned}\right\} \tag{30}$$

summed for $i = 0, 2, 4, \ldots, (2N-2)$, must satisfy the boundary conditions (3) and (4) at θ_j ($j = 0, 2, 4, \ldots, (2N-2)$). Let $\mathscr{R}_j$ stand for the value of $\mathscr{R}$ along $\theta = \theta_j$; and let us, for brevity in printing, use P_{ij} for the Legendre polynomial $P_i(\cos\theta_j)$. Similarly, let P_{ij}^1 denote the value of

$$P_i^1(\cos\theta_j) = -[(d/d\theta)\,P_i(\cos\theta)]_{\theta=\theta_j}.$$

Also, define functions Q_{ij} such that

$$S_j = \mathscr{R}_i Q_{ij}, \tag{31}$$

i.e. since

$$\mathscr{R}_j = S_i P_{ij}, \tag{32}$$

$$P_{li} Q_{ij} = \delta_{lj}, \tag{33}$$

where δ_{lj} is the Kronecker delta. In the last three equations, and henceforward, the summation convention is used with respect to all repeated indices except the current index j. Then, substituting these expressions in the boundary conditions (3) and (4), the following system of $3N$ equations in the $3N$ variables A_j, B_j, $\mathscr{R}_j$ is obtained:

$$\begin{aligned}\mathscr{R}_l Q_{li} P_{ij}^1 A_k \mathscr{R}_j^{k-2} P_{kj}^1 - iA_i \mathscr{R}_j^{i-1} P_{ij} &= d\mathscr{R}_j/d\tau\\ &= \mathscr{R}_l Q_{li} P_{ij}^1 B_k \mathscr{R}_j^{-k-3} P_{kj}^1 + (i+1)\,B_i \mathscr{R}_j^{-i-2} P_{ij},\end{aligned} \tag{34}$$

$$\begin{aligned}(1+\epsilon)\,(dA_i/d\tau)\,\mathscr{R}_j^i P_{ij} - \tfrac{1}{2}(1+\epsilon)\,[(iA_i\mathscr{R}_j^{i-1}P_{ij})^2 + (A_i\mathscr{R}_j^{i-1}P_{ij}^1)^2] - \epsilon\mathscr{R}_j\cos\theta_j&\\ = (dB_i/d\tau)\,\mathscr{R}_j^{-i-1}P_{ij} - \tfrac{1}{2}[(i+1)\,B_i\mathscr{R}_j^{-i-2}P_{ij}]^2 - \tfrac{1}{2}(B_i\mathscr{R}_j^{-i-2}P_{ij}^1)^2&\end{aligned} \tag{35}$$

for $i, j, k, l = 0, 2, 4, \ldots, (2N-2)$. As before, ϵ stands for $(\rho_1 - \rho_2)/\rho_2$.

The numerical solution of this system of $3N$ equations in general necessitates the simultaneous solution of $2N$ equations with constant coefficients at each small interval of time. But, in the particular case of a fluid column *in vacuo*, or approximately, when the density of the fluid column is large compared with that of the surrounding medium, $\epsilon \to \infty$, and the system reduces to the $2N$ equations

$$d\mathscr{R}_j/d\tau = \mathscr{R}_l Q_{li} P_{ij}^1 A_k \mathscr{R}_j^{k-2} P_{kj}^1 - iA_i \mathscr{R}_j^{i-1} P_{ij}, \tag{36}$$

and

$$\mathscr{R}_j^i P_{ij}\, dA_i/d\tau = \tfrac{1}{2}[(iA_i\mathscr{R}_j^{i-1}P_{ij})^2 + (A_i\mathscr{R}_j^{i-1}P_{ij}^1)^2] + \mathscr{R}_j\cos\theta_j, \tag{37}$$

of which the numerical solution necessitates the simultaneous solution of no more than N equations with constant coefficients at each small interval of time.

In the case of plane symmetry, similar finite systems of equations are obtained involving $\cos i\theta_j$, $\sin i\theta_j$ instead of P_{ij}, P_{ij}^1.

As a particular example, the full system of eight equations may be quoted for the plane symmetric case of an initially hemi-cylindrical column resting on its diametral plane, when $N = 4$ and the interface conditions are satisfied at $\theta_j = \frac{1}{12}j\pi$ $(j = 0, 2, 4, 6)$. These are

$$\left.\begin{aligned}
\frac{d\mathscr{R}_0}{d\tau} &= -2A_2\mathscr{R}_0 - 4A_4\mathscr{R}_0^3 - 6A_6\mathscr{R}_0^5,\\
\frac{d\mathscr{R}_2}{d\tau} &= -A_2\mathscr{R}_2 + 2A_4\mathscr{R}_2^3 + 6A_6\mathscr{R}_2^5 + (A_2 + 2A_4\mathscr{R}_2^2)\,(3\mathscr{R}_0 - \mathscr{R}_2 - 3\mathscr{R}_4 + \mathscr{R}_6),\\
\frac{d\mathscr{R}_4}{d\tau} &= \quad A_2\mathscr{R}_4 + 2A_4\mathscr{R}_4^3 - 6A_6\mathscr{R}_4^5 + (A_2 - 2A_4\mathscr{R}_4^2)\,(-\mathscr{R}_0 + 3\mathscr{R}_2 + \mathscr{R}_4 - 3\mathscr{R}_6),\\
\frac{d\mathscr{R}_6}{d\tau} &= \quad 2A_2\mathscr{R}_6 - 4A_4\mathscr{R}_6^3 + 6A_6\mathscr{R}_6^5,
\end{aligned}\right\} \tag{38}$$

and

$$\left.\begin{aligned}
&\frac{dA_0}{d\tau} + \mathscr{R}_0^2\frac{dA_2}{d\tau} + \mathscr{R}_0^4\frac{dA_4}{d\tau} + \mathscr{R}_0^6\frac{dA_6}{d\tau}\\
&\quad = 2A_2^2\mathscr{R}_0^2 + 8A_4^2\mathscr{R}_0^6 + 18A_6^2\mathscr{R}_0^{10} + 8A_2A_4\mathscr{R}_0^4 + 24A_4A_6\mathscr{R}_0^8 + 12A_6A_2\mathscr{R}_0^6 + \mathscr{R}_0,\\
&\frac{dA_0}{d\tau} + \tfrac{1}{2}\mathscr{R}_2^2\frac{dA_2}{d\tau} - \tfrac{1}{2}\mathscr{R}_2^4\frac{dA_4}{d\tau} - \mathscr{R}_2^6\frac{dA_6}{d\tau}\\
&\quad = 2A_2^2\mathscr{R}_2^2 + 8A_4^2\mathscr{R}_2^6 + 18A_6^2\mathscr{R}_2^{10} + 4A_2A_4\mathscr{R}_2^4 + 12A_4A_6\mathscr{R}_2^8 - 6A_6A_2\mathscr{R}_2^6 + \sqrt{3}\mathscr{R}_2/2,\\
&\frac{dA_0}{d\tau} - \tfrac{1}{2}\mathscr{R}_4^2\frac{dA_2}{d\tau} - \tfrac{1}{2}\mathscr{R}_4^4\frac{dA_4}{d\tau} + \mathscr{R}_4^6\frac{dA_6}{d\tau}\\
&\quad = 2A_2^2\mathscr{R}_4^2 + 8A_4^2\mathscr{R}_4^6 + 18A_6^2\mathscr{R}_4^{10} - 4A_2A_4\mathscr{R}_4^4 - 12A_4A_6\mathscr{R}_4^8 - 6A_6A_2\mathscr{R}_4^6 + \tfrac{1}{2}\mathscr{R}_4,\\
&\frac{dA_0}{d\tau} - \mathscr{R}_6^2\frac{dA_2}{d\tau} + \mathscr{R}_6^4\frac{dA_4}{d\tau} - \mathscr{R}_6^6\frac{dA_6}{d\tau}\\
&\quad = 2A_2^2\mathscr{R}_6^2 + 8A_4^2\mathscr{R}_6^6 + 18A_6^2\mathscr{R}_6^{10} - 8A_2A_4\mathscr{R}_6^4 - 24A_4A_6\mathscr{R}_6^8 + 12A_6A_2\mathscr{R}_6^6.
\end{aligned}\right\} \tag{39}$$

In order to illustrate the effect of using the expansions (10a) instead of (10) we also give the corresponding equations using (10a). The equations (39) are not altered, but instead of (38) we have the following set:

$$\left.\begin{aligned}
\frac{d\mathscr{R}_0}{d\tau} &= -2A_2\mathscr{R}_0 - 4A_4\mathscr{R}_0^3 - 6A_6\mathscr{R}_0^5,\\
\frac{d\mathscr{R}_2}{d\tau} &= -A_2\mathscr{R}_2 + 2A_4\mathscr{R}_2^3 + 6A_6\mathscr{R}_2^5\\
&\qquad + (A_2 + 2A_4\mathscr{R}_2^2)\,(4{\cdot}098\mathscr{R}_0 - 3{\cdot}098\mathscr{R}_2 - 1{\cdot}098\mathscr{R}_4 + 0{\cdot}098\mathscr{R}_6),\\
\frac{d\mathscr{R}_4}{d\tau} &= \quad A_2\mathscr{R}_4 + 2A_4\mathscr{R}_4^3 - 6A_6\mathscr{R}_4^5\\
&\qquad - (A_2 - 2A_4\mathscr{R}_4^2)\,(4{\cdot}098\mathscr{R}_0 - 8{\cdot}830\mathscr{R}_2 + 4{\cdot}098\mathscr{R}_4 + 0{\cdot}634\mathscr{R}_6),\\
\frac{d\mathscr{R}_6}{d\tau} &= \quad 2A_2\mathscr{R}_6 - 4A_4\mathscr{R}_6^3 + 6A_6\mathscr{R}_6^5.
\end{aligned}\right\} \tag{38a}$$

Numerical solutions both in plane and axial symmetry and for different values of N are given in the following paragraphs.

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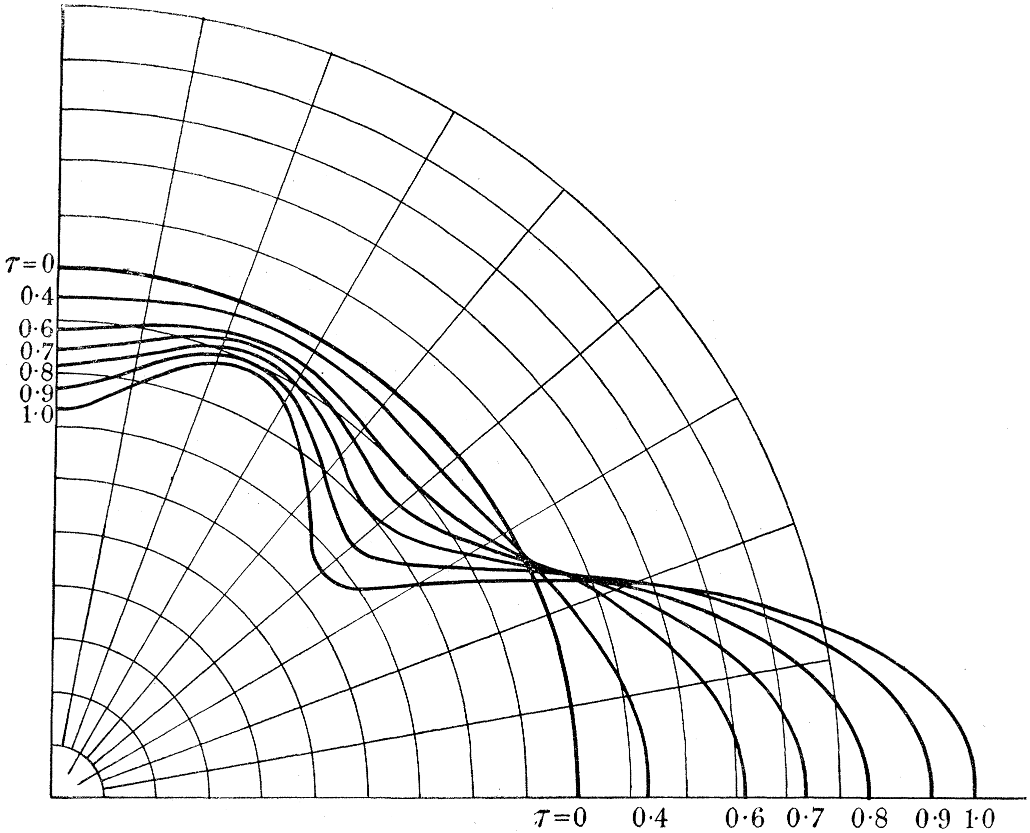

FIGURE 4. Collapse of the plane symmetric column *in vacuo*, initially semicircular at rest; $N = 4$.

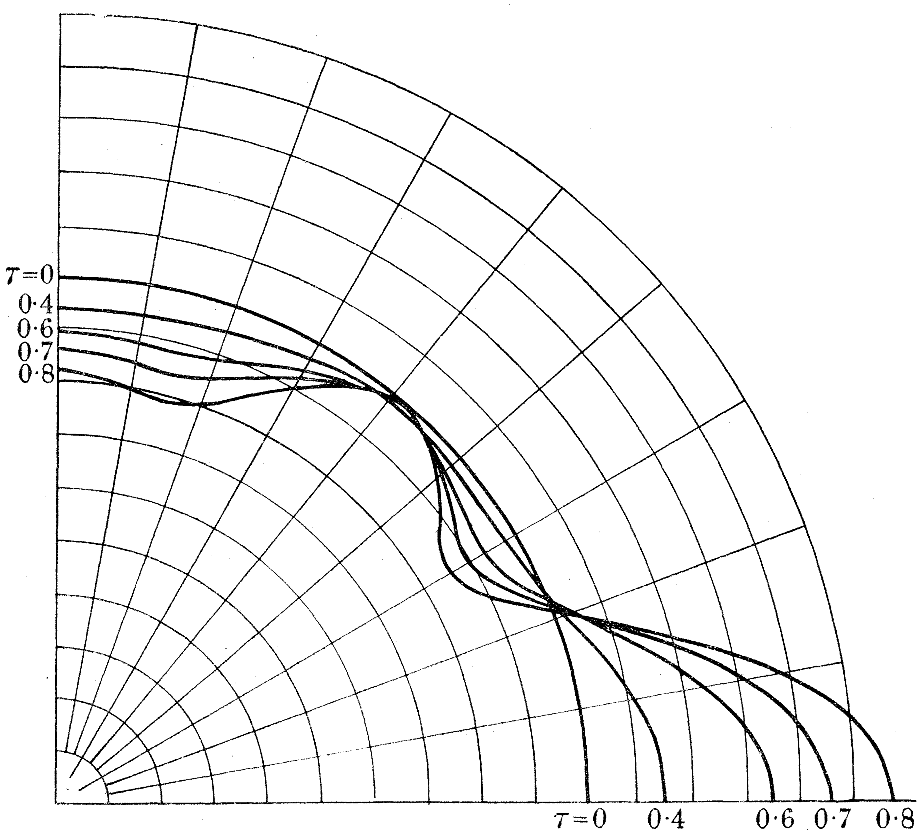

FIGURE 5. Collapse of the plane symmetric column *in vacuo*, initially semicircular at rest; $N = 5$.

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Various numerical examples. Figure 4 shows the approximate initial motion for a plane symmetric column of initially semicircular cross-section *in vacuo*, as calculated from the equations (38) and (39), i.e. for $N = 4$ and satisfying the interface conditions at $\theta_j = \frac{1}{12}j\pi$ $(j = 0, 2, 4, 6)$.

Figure 5 shows the corresponding solution in which $N = 5$, and the interface conditions are satisfied at $\theta_j = \frac{1}{16}j\pi$ $(j = 0, 2, 4, 6, 8)$. A comparison with figure 4 shows the influence of the choice of N and the angles θ_j on the numerical solution at these small values of N. The general shape of the columns at corresponding times and the general nature of the motion is the same in both cases. Detailed differences due to the choice of N and θ_j are clearly seen.

One would expect from physical arguments that the mean of the solutions with $N = 4$ and $N = 5$ would be a better approximation than either. By taking the average, the detailed oscillatory differences due to the choice of N and θ_j do very nearly balance out, and a smooth column boundary without extraneous oscillations is left (cf. figure 13).

The duration in time for which the above solutions have any practical significance may be assessed by the extent to which they satisfy the principles of conservation of mass and energy. The ratios of the computed mass and energy at any stage expressed in terms of their original values are given for these solutions in table 2.

TABLE 2

(Plane symmetry; $\mathscr{R}$ expanded in even cosines)

	τ	0	0·2	0·4	0·6	0·8	1·0
$N=4$ (figure 4)	mass factor	1	1·001	1·005	1·022	1·054	1·059
$N=4$ (figure 4)	energy factor	1	1·006	1·012	1·039	1·111	1·204
$N=5$ (figure 5)	mass factor	1	1·000	1·003	1·017	1·031	—

Figure 6 shows the corresponding solution in axial symmetry for a column, initially hemispherical, *in vacuo*, in which $N = 4$ and the interface conditions are satisfied at $\theta_j = \frac{1}{12}j\pi$ $(j = 0, 2, 4, 6)$. As compared with the plane symmetric solution of figure 4, the axially symmetric column spreads out more slowly over the base-plane, but loses height more rapidly, as would be expected. The axially symmetric solution also diverges more rapidly in time from the principles of conservation of mass and energy than the corresponding plane solution of figure 4. For comparison with table 2, the mass factors for the axially symmetric solution are given in table 3.

TABLE 3

(Axial symmetry; $\mathscr{R}$ expanded in even harmonics)

	τ	0	0·2	0·4	0·6	0·7
$N=4$ (figure 6)	mass factor	1	1·001	1·011	1·057	1·111

For any of these numerical solutions it is possible to calculate the pressure distribution at any time by means of equation (1). It is of particular interest to compare the initial pressure distribution calculated in this way with the known initial pressure distribution on the column boundary. For example, in the case of a fluid column *in vacuo*, the supporting pressure at the boundary point (R, θ), just before the fluid is released, must be $\rho g(h - \mathrm{R}\cos\theta)$, and the pressure on the horizontal supporting plane is ρgh, where h is the maximum height

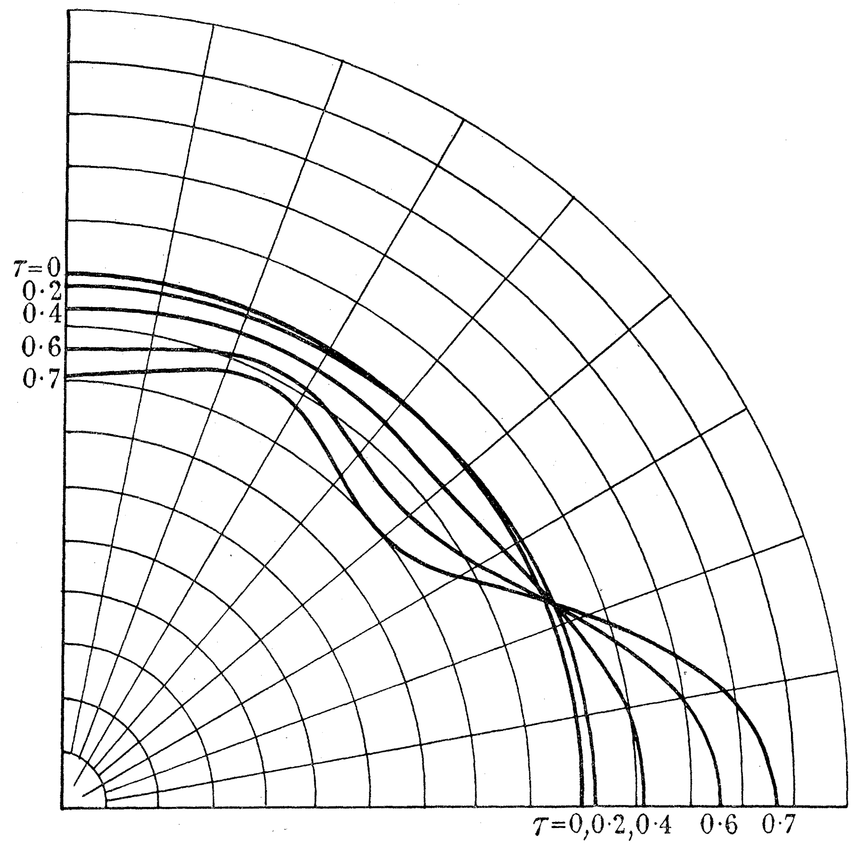

FIGURE 6. Collapse of the axially symmetric column *in vacuo*, initially hemispherical at rest; $N = 4$.

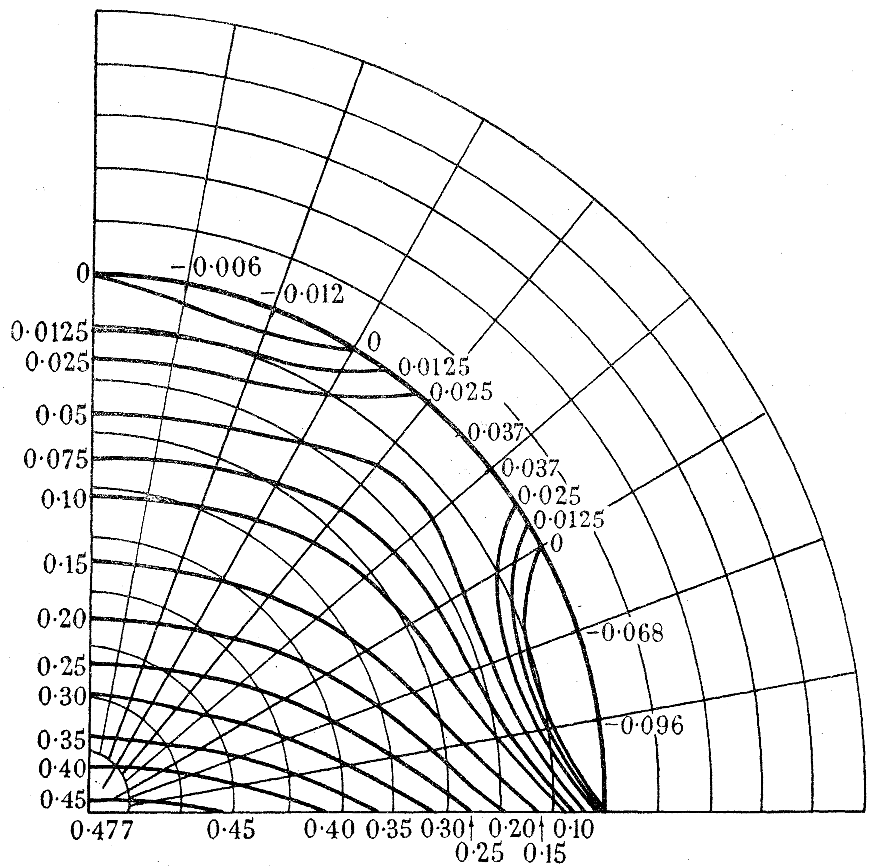

FIGURE 7. The initial distribution of pressure in the fluid column shown in figure 4. The isobars are drawn.

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of the column. At the instant of release, however, the column boundary must become the isobar $p = 0$, and the pressure on the horizontal supporting plane will be reduced non-uniformly on account of the instantaneous accelerations set up in the column. Figure 7 shows the calculated distribution of the isobars $p/\rho ga =$ constant, at $\tau = 0$ for the initially hemispherical column *in vacuo*, with $N = 4$ (cf. figure 6). Even with such a small value of N the calculated values of $p/\rho ga$ on the column boundary are very small, except near $\theta = \frac{1}{2}\pi$. At the centre of the base of the column, the pressure, which is ρga just before the column is released, falls instantaneously to $0{\cdot}477\rho ga$. In the corresponding plane symmetric case (cf. figures 4, 5), the initial distribution of isobars is very similar, and the values of $p/\rho ga$ on the column boundary are smaller with $N = 4$, and smaller still, of course, with $N = 5$. The calculated pressure at the centre of the base of the column, at the instant of release, is $0{\cdot}622\rho ga$ with $N = 4$, and $0{\cdot}628\rho ga$ with $N = 5$.

5. Approximate solution for squat columns

At any stage of the motion when the maximum surviving height of the column is small compared with its base diameter, vertical accelerations in the motion will be correspondingly small compared with horizontal accelerations. In such circumstances, an interesting type of approximate solution may be derived by making the sole assumption, following Lamb's *Hydrodynamics*, art. 187, that vertical acceleration may be neglected, or more precisely, that the pressure at any point is sensibly equal to the static pressure due to its depth below the free surface.

First, we shall deal with the case where the column rests on a rigid horizontal plane, but is not partially supported by a second liquid. Finally, we deal with the case where the second liquid is also present.

Second fluid absent

Let co-ordinates (y, z) be chosen, with y measured vertically upwards from the base-plane, and z measured horizontally from the plane (or radially from the axis) of symmetry; and let the height of the free surface at any point be $\eta(z, t)$.

Then the condition of zero pressure at the free surface gives, on the above assumption,

$$p = \rho g(\eta - y),$$

so that

$$\frac{\partial p}{\partial z} = \rho g \frac{\partial \eta}{\partial z},$$

which is independent of y. The horizontal (or radial) acceleration of the fluid is thus the same for all particles which are equidistant from the plane (or axis) of symmetry. It follows that all particles which are so equidistant at any time will remain so; in other words, the horizontal (or radial) velocity u is a function of z and t only.

Then, of the two equations of conservation of momentum, namely,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial z},$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial z} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g,$$

the second is satisfied identically by virtue of the assumption of negligible vertical acceleration, and the first reduces to

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial z}=-g\frac{\partial \eta}{\partial z}. \tag{40}$$

The boundary condition on velocity at the free surface is

$$\int_0^\eta \frac{\partial v}{\partial y}dy=\frac{\partial \eta}{\partial t}+u\frac{\partial \eta}{\partial z},$$

and this combines with the equation of conservation of mass,

$$\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}+\frac{ku}{z}=0,$$

to give

$$\frac{\partial \eta}{\partial t}+u\frac{\partial \eta}{\partial z}+\frac{ku\eta}{z}=-\eta\frac{\partial u}{\partial z}. \tag{41}$$

In the above equations, k is a constant, with the value zero for plane symmetry, and unity for axial symmetry.

From (40) and (41), it follows that, for any λ,

$$\left[\frac{\partial}{\partial t}+(u+\lambda\eta)\frac{\partial}{\partial z}\right]u+\lambda\left[\frac{\partial}{\partial t}+\left(u+\frac{g}{\lambda}\right)\frac{\partial}{\partial z}\right]\eta+\frac{\lambda ku\eta}{z}=0, \tag{42}$$

and the two operators in this equation may be made identical by choosing λ to satisfy

$$\lambda\eta=g/\lambda,$$

i.e.

$$\lambda=\pm(g/\eta)^{\frac{1}{2}}.$$

In this case, equation (42) reduces to

$$\left[\frac{\partial}{\partial t}+\{u\pm(g\eta)^{\frac{1}{2}}\}\frac{\partial}{\partial z}\right][u\pm2(g\eta)^{\frac{1}{2}}]\pm\frac{ku(g\eta)^{\frac{1}{2}}}{z}=0. \tag{43}$$

The problem has thus been reduced to one of the hyperbolic type in the two independent variables (z, t). The characteristic curves are given by

$$\frac{dz}{dt}=u\pm(g\eta)^{\frac{1}{2}}, \tag{44}$$

and along these curves hold respectively the relations

$$u\pm2(g\eta)^{\frac{1}{2}}\pm k\int^z\frac{u(g\eta)^{\frac{1}{2}}}{[u\pm(g\eta)^{\frac{1}{2}}]}\frac{dz}{z}=\text{constant}. \tag{45}$$

The remaining boundary conditions for a column starting from rest are

$$\left.\begin{aligned}
z=0:&\quad u=0\quad\text{for all } t,\\
t=0:&\quad u=0\quad\text{for all } z,\\
&\quad\eta=an^2[1-f_0(z/a)]\quad\text{for } |z|\leqslant a \text{ and } f_0(1)=1,\\
&\quad\eta=0\quad\text{for } |z|>a.
\end{aligned}\right\} \tag{46}$$

Here $2a$ is the initial base measurement of the column, n^2a its initial height, and $f_0(z/a)$ is a function defining the shape of its initial cross-section.

Non-dimensional variables

With the transformation
$$\left.\begin{aligned} Z &= z/a, \\ H &= \eta/an^2, \\ T &= nt(g/a)^{\frac{1}{2}}, \\ U &= u/n(ag)^{\frac{1}{2}}, \end{aligned}\right\} \tag{47}$$

the characteristic curves become
$$\frac{dZ}{dT} = U \pm H^{\frac{1}{2}}, \tag{48}$$

along which hold respectively the relations

$$U \pm 2H^{\frac{1}{2}} \pm k \int^{Z} \frac{UH^{\frac{1}{2}}}{(U \pm H^{\frac{1}{2}})} \frac{dZ}{Z} = \text{constant}, \tag{49}$$

and the boundary conditions reduce to

$$\left.\begin{aligned} Z = 0: \quad & U = 0 \quad \text{for all } T, \\ T = 0: \quad & U = 0 \quad \text{for all } Z, \\ & H = 1 - f_0(Z) \quad \text{for} \quad |Z| \leqslant 1 \text{ and } f_0(1) = 1, \\ & H = 0 \quad \text{for} \quad |Z| > 1. \end{aligned}\right\} \tag{50}$$

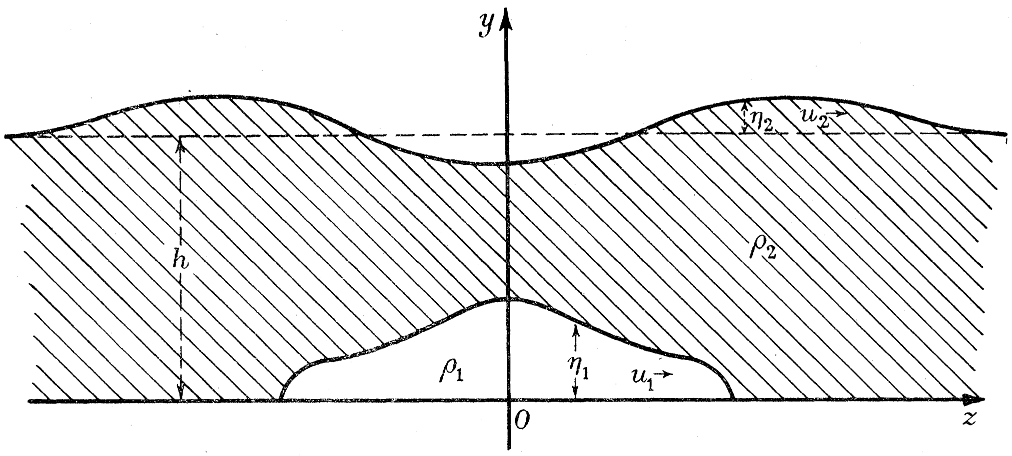

Figure 8. Notation used for the motion of a squat column of liquid 1 at the bottom of a sheet of liquid 2, of average depth h.

Fluid column partially supported by a second fluid

It is also possible, with the assumption of negligible vertical acceleration, to deal with the problem of a column immersed in another fluid. Consider a fluid column immersed in an outer fluid of finite mean depth h, and let suffix 1 refer to the fluid inside the column, and suffix 2 to the surrounding fluid.

The equation of conservation of momentum in fluid 1 is then

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial z} = -\frac{1}{\rho_1} \frac{\partial}{\partial z} [\rho_2 g(h + \eta_2 - \eta_1) + \rho_1 g(\eta_1 - y)]. \tag{51}$$

Writing $\epsilon = \rho_1/\rho_2 - 1$, this becomes

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial z} = -\frac{\epsilon g}{1+\epsilon} \frac{\partial \eta_1}{\partial z} - \frac{g}{1+\epsilon} \frac{\partial \eta_2}{\partial z}. \tag{52}$$

Similarly, the equation of conservation of momentum in fluid 2 is

$$\frac{\partial u_2}{\partial t}+u_2\frac{\partial u_2}{\partial z}=-g\frac{\partial \eta_2}{\partial z}. \tag{53}$$

The equations of conservation of mass are

$$\frac{\partial \eta_1}{\partial t}+u_1\frac{\partial \eta_1}{\partial z}+\eta_1\frac{\partial u_1}{\partial z}+\frac{k\eta_1 u_1}{z}=0 \tag{54}$$

and
$$\frac{\partial}{\partial t}(\eta_2-\eta_1)+u_2\frac{\partial}{\partial z}(\eta_2-\eta_1)+(h+\eta_2-\eta_1)\frac{\partial u_2}{\partial z}+(ku_2/z)\,(h+\eta_2-\eta_1)=0. \tag{55}$$

Now h only occurs in equation (55), and if $h\to\infty$ this equation reduces to

$$\frac{1}{z}\frac{\partial(zu_2)}{\partial z}=0 \quad \text{or} \quad \frac{\partial u_2}{\partial z}=0,$$

according as $k=1$ or 0, i.e. zu_2 or u_2 respectively is a function of t only. But the condition for symmetrical motion is

$$z=0: \quad u_2=0 \text{ for all } t,$$

and hence, in this case, $u_2=0$ for all z and t. It follows from (53) then, that $\partial\eta_2/\partial z=0$, and (52) and (54) thus reduce to

$$\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial z}=-\frac{\epsilon g}{1+\epsilon}\frac{\partial \eta_1}{\partial z}, \tag{56}$$

$$\frac{\partial \eta_1}{\partial t}+u_1\frac{\partial \eta_1}{\partial z}+\eta_1\frac{\partial u_1}{\partial z}+\frac{k\eta_1 u_1}{z}=0. \tag{57}$$

Non-dimensional variables

The transformation

$$\left.\begin{aligned} Z&=z/a,\\ H&=\eta/an^2,\\ T&=nt/\{a(1+\epsilon)/\epsilon g\}^{\frac{1}{2}},\\ U&=u/n\{a\epsilon g/(1+\epsilon)\}^{\frac{1}{2}},\end{aligned}\right\} \tag{58}$$

then again reduces the problem and the boundary conditions to the forms given in (48), (49) and (50).

The approximate solution outlined above is thus one which can be readily computed by the numerical method of characteristics. The transformation (47) shows that the dispersive motion of all columns *in vacuo* which have the same initial shape except for a scale factor in height, depend, under the assumption of negligible vertical acceleration, on the same characteristic solution. The more general transformation (58) shows further that the same result is also true when the column is immersed in another fluid which extends upwards and outwards to infinity, differences in the relative densities of the two fluids being equivalent to a reduction in the value of gravity by a factor $\epsilon/(1+\epsilon)$.

Numerical solutions, obtained by this method, are given in the following section.

6. Calculations for squat columns

Several calculations have been made by the numerical method of characteristics, neglecting vertical accelerations in the motion *ab initio*. For the plane symmetric case of a column initially at rest with a rectangular cross-section (cf. equations (50))

$$\left.\begin{aligned} f_0(Z) &= 0 \quad \text{for} \quad |Z|<1 \\ &= 1 \quad \text{for} \quad |Z|=1. \end{aligned}\right\} \tag{59}$$

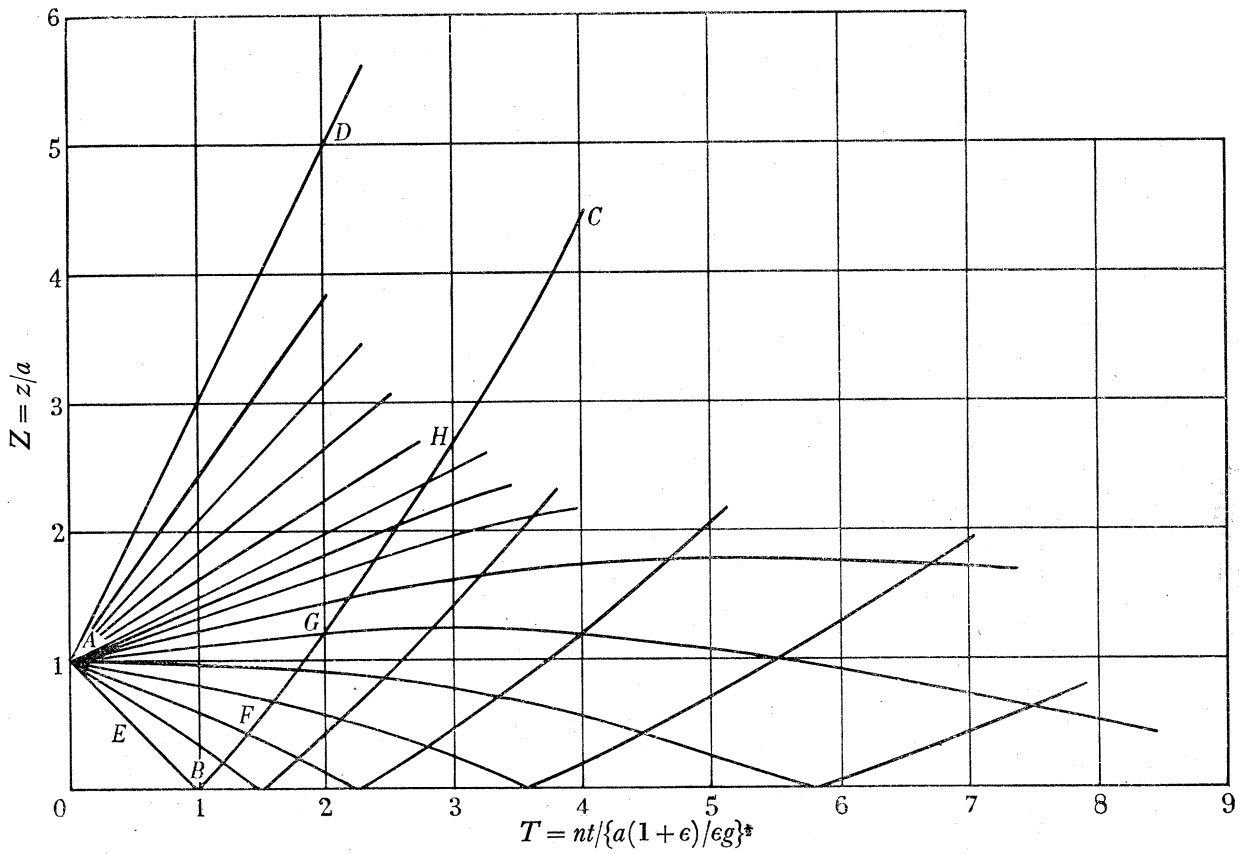

Figure 9. Characteristic diagram for the plane symmetric column, initially rectangular at rest, and of height $\frac{1}{2}n^2$ times its width.

Figure 9 shows the characteristic diagram in this case for one-half of the symmetric motion, and figure 10 shows the corresponding motion of the column boundary. The motion consists of two simple waves, centred respectively on A and its reflexion A' in $Z=0$, which interact when they both reach the centre of the column at B. The ultimate characteristic AD of the simple wave about A is $Z=2T$, and this gives the physical result that the base of the column travels outwards, *ab initio*, with the constant specific velocity $U=2$. It can easily be shown that the non-linear characteristic through any point

$$Z = 1-\alpha,$$

$$T = \alpha$$

on AB is the cubic

$$(2T+1-Z)^3 = 27\alpha^2 T. \tag{60}$$

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Thus the characteristic *BFGH ... C*, which terminates the simple wave centred on *A*, is the cubic

$$(2T+1-Z)^3 = 27T. \tag{61}$$

Within the simple wave *DABC*

$$\left.\begin{aligned} U &= 2(Z-1+T)/3T, \\ H &= [(2T-Z+1)/3T]^2, \end{aligned}\right\} \tag{62}$$

and thus, at any given time, the distribution of H across the simple wave *DABC* is parabolic. It is of interest to note that the characteristic geometry corresponds to that of the steady supersonic flow of a compressible fluid through an orifice into a vacuum.

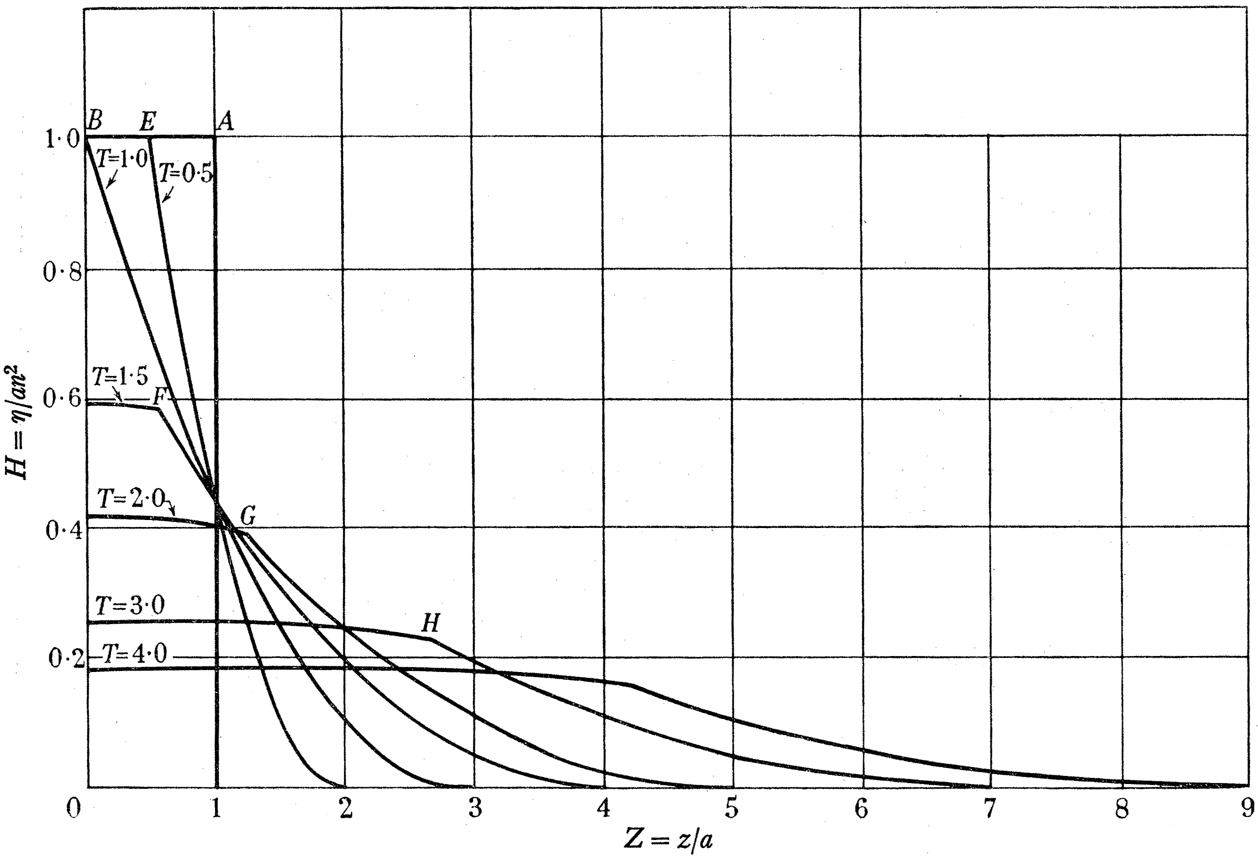

FIGURE 10. Collapse of a rectangular column, initially at rest and of height $\frac{1}{2}n^2$ times its width.

Figures 11 and 12 show the corresponding solution for a plane symmetric column initially at rest with a semi-elliptic cross-section, i.e.

$$f_0(Z) = 1-(1-Z^2)^{\frac{1}{2}}. \tag{63}$$

In this case, the base of the column accelerates from zero velocity up to the terminal value $U = 2$.

Figure 13 shows the comparison between this solution for the column boundary at $T = 0{\cdot}8$ in the particular case ($n = 1$, $\epsilon \to \infty$) of an initially semicircular column *in vacuo*, and the corresponding solution derived by the method of § 4 above.

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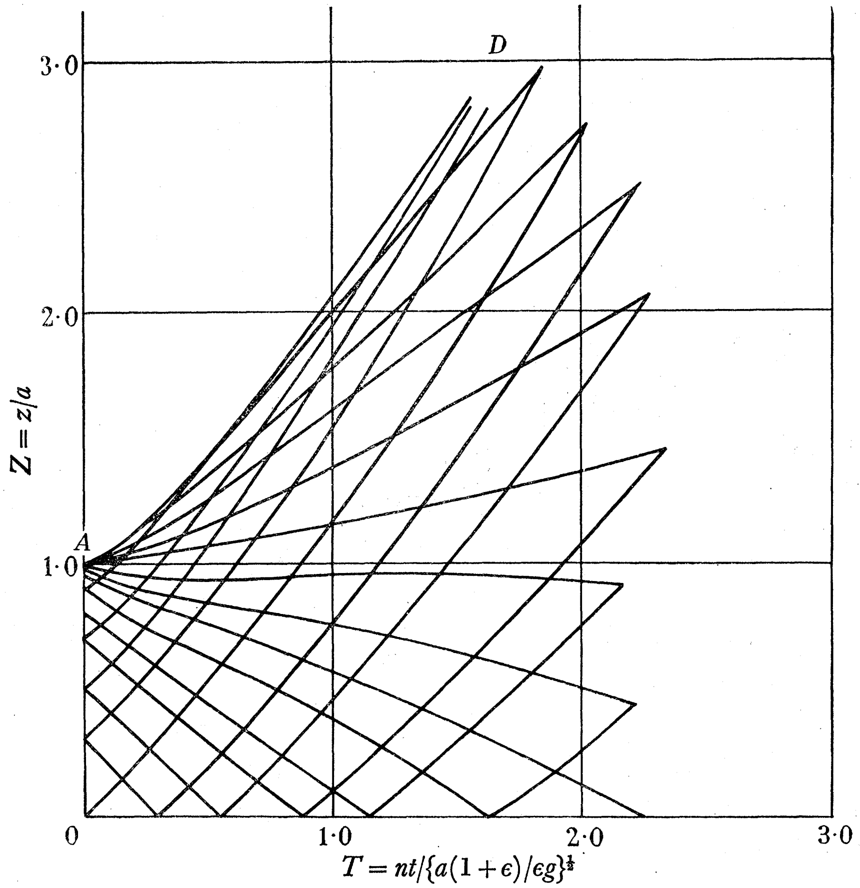

FIGURE 11. Characteristic diagram for a plane symmetric column initially at rest with a semi-elliptic cross-section. The initial height at the centre of the column (the minor semi-axis of the ellipse) is $\frac{1}{2}n^2$ times the initial width at the base (the major axis of the ellipse).

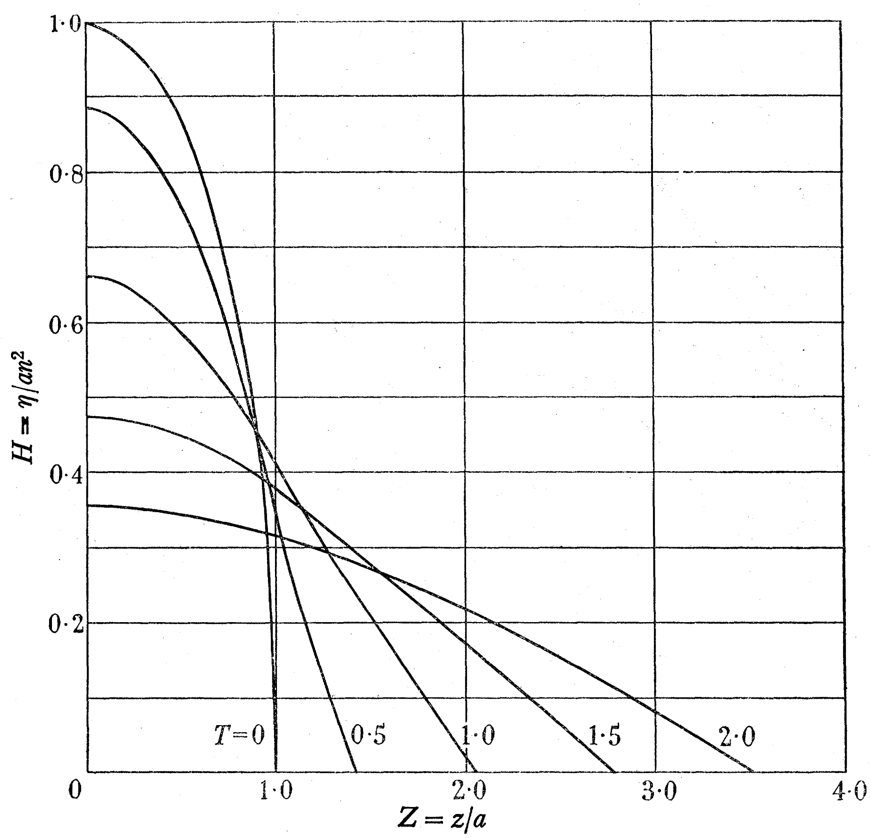

FIGURE 12. Collapse of the semi-elliptic column described in figure 11.

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Figures 14 and 15 show the results obtained for an axially symmetric column initially at rest with a rectangular cross-section. In the case of axial symmetry, the integral term in the relations (49) which hold along the characteristics has an integrand whose numerator and denominator both tend to zero as the axis $Z = 0$ is approached. This difficulty is sur-

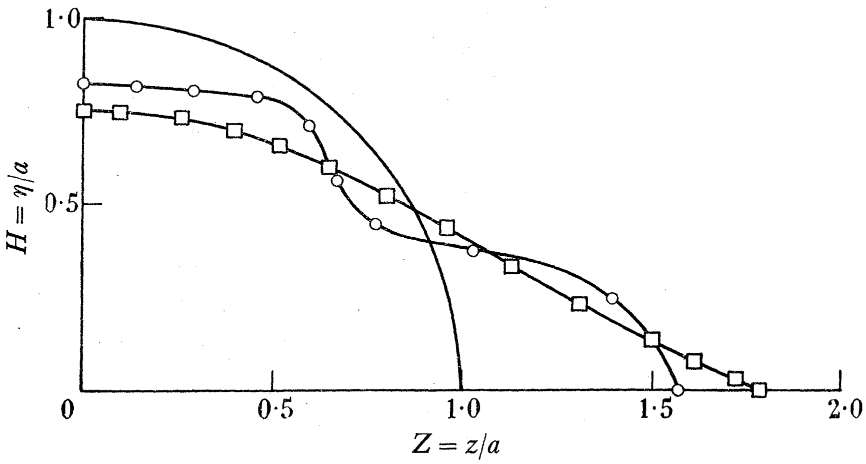

FIGURE 13. A comparison of the shape of a column *in vacuo*, initially hemi-cylindrical at rest, at specific time $T = 0{\cdot}8$. The curve through the circles is the average of the solutions with $N = 4$ and $N = 5$ shown in figures 4 and 5; and the curve through the squares is the solution obtained by the method of characteristics (i.e. neglecting vertical acceleration).

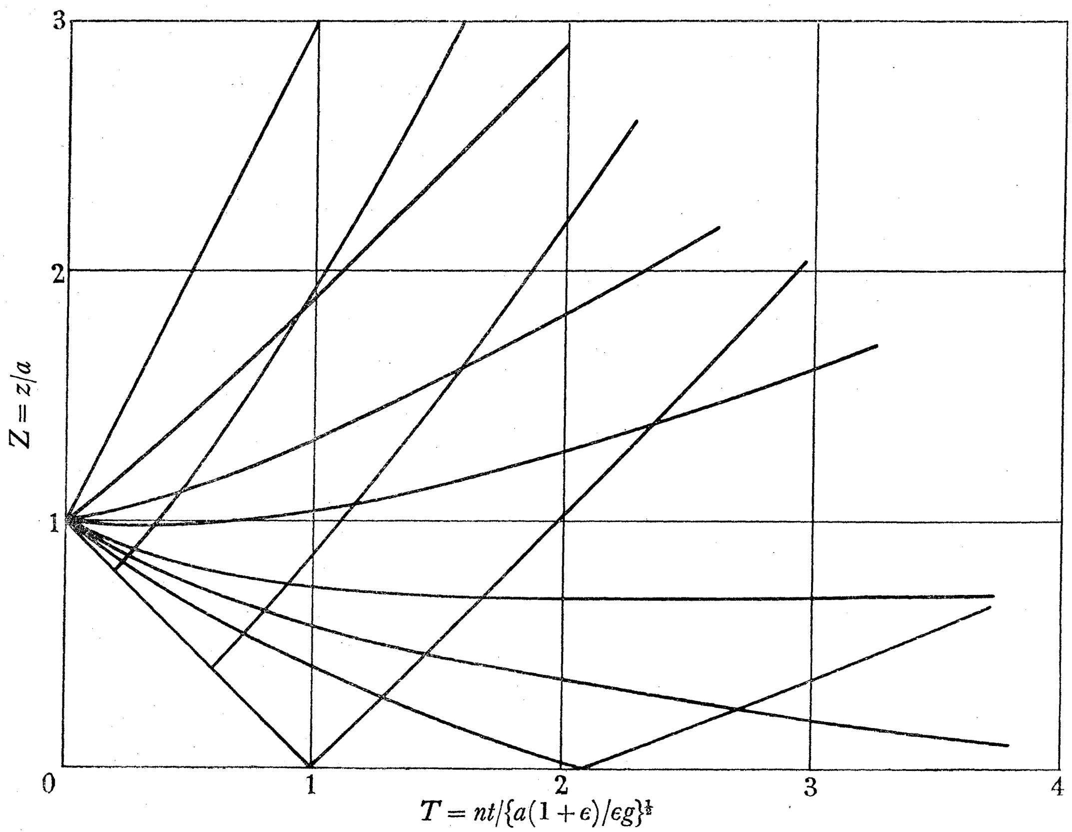

FIGURE 14. Characteristic diagram for an axially symmetric column in a second fluid, initially rectangular in cross-section of height $\frac{1}{2}n^2$ times the width.

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mounted in the numerical calculations by making use, in the neighbourhood of the axis of symmetry, of the relation

$$\lim_{Z\to 0}\frac{UH^{\frac{1}{2}}}{Z(U\pm H^{\frac{1}{2}})}=\mp\frac{1}{H_0^{\frac{1}{2}}}\frac{dH_0^{\frac{1}{2}}}{dT},\tag{64}$$

where H_0 denotes the value of H on the axis of symmetry. A more complicated process of iteration is thus necessitated in the small arcs of computation adjacent to the central axis. This process involves extrapolation of the gradient of $H^{\frac{1}{2}}$ along this axis.

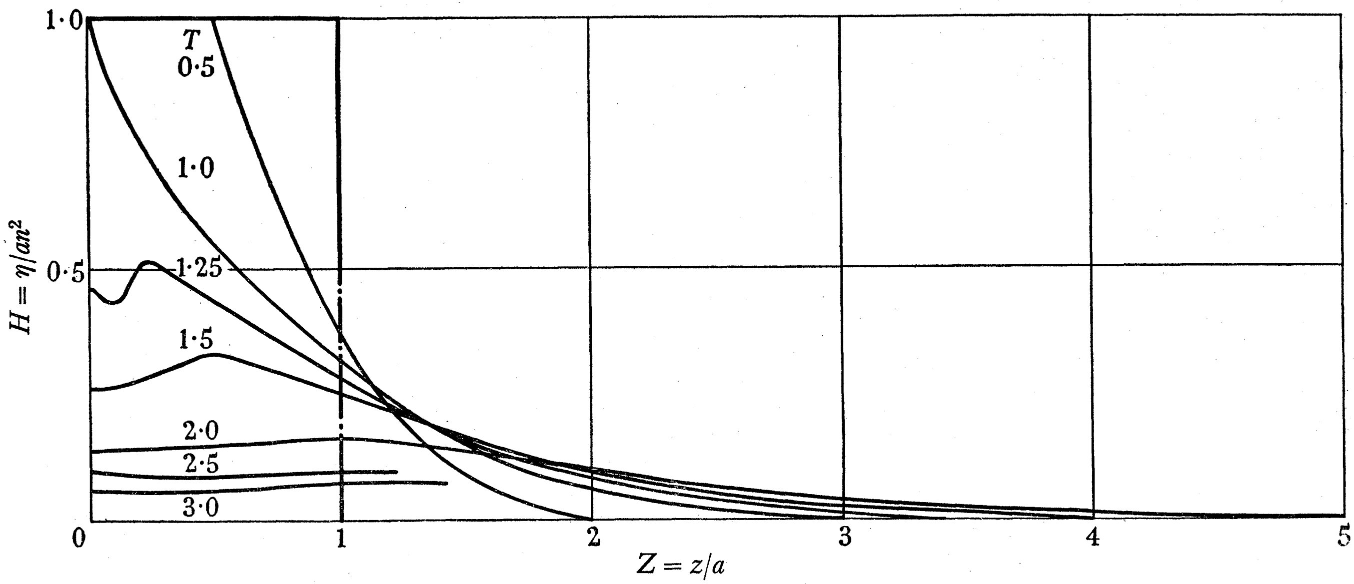

FIGURE 15. The shape at various times of the column described in figure 14.

It is a pleasure to record that the calculations described in § 4 were performed entirely by Miss K. M. Stocks, and those in § 6 by Mr P. L. Owen, Miss P. M. E. Martin and Miss D. M. Jones. To all of these we express our thanks for their co-operation. We also wish to thank Professor Harold Jeffreys and Mr C. H. Kebby for helpful discussions on some of the points which have arisen in the preparation of this paper.

We are indebted to the Chief Scientist, Ministry of Supply, for permission to publish this paper.

7. APPENDIX

By L. FOX and E. T. GOODWIN

National Physical Laboratory

An alternative method of solving the general problem was suggested and tried by the Mathematics Division, National Physical Laboratory.

This method attempts to calculate values of the velocity potential, and the position of the moving fluid boundary, by a direct numerical attack on the governing differential equations and boundary conditions.

In the non-dimensional form produced by equations (12) it is required to solve, in the plane case, the differential equation

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0,\tag{A 1}$$

where Φ is also a function of τ. On the boundary, with no outer fluid, there is the pressure condition

$$\frac{\partial \Phi}{\partial \tau}=\frac{1}{2}\left[\left(\frac{\partial \Phi}{\partial x}\right)^2+\left(\frac{\partial \Phi}{\partial y}\right)^2\right]+y, \tag{A 2}$$

and a second condition that the normal velocity of the fluid is equal to that of the boundary.

The suggested procedure is to use finite steps in time, calculating at successive instants the velocity potential at points, located at nodes of a square mesh, lying within the fluid. If at any time τ the position of the boundary and the values of $\Phi(\tau)$ on it are known, $\Phi(\tau)$ at internal points can be calculated by use of equation (A 1) and relaxation methods. The boundary conditions are then used to give an estimate of the boundary position at time $\tau+\delta\tau$, and the new values $\Phi(\tau+\delta\tau)$ on this boundary. In theory the process can be extended in time as far as desired.

It is not difficult to produce finite-difference equations, for Φ and boundary position, which are satisfactory for small and well-behaved variations of these quantities. If the suffixes 1, 0 and -1 refer to conditions at times $\tau+\delta\tau$, τ and $\tau-\delta\tau$ respectively, and if X, Y are the co-ordinates, along particular horizontal and vertical mesh lines, of the boundary points, the following equations can be deduced from the boundary conditions:

$$X_1-X_{-1}=-2\delta\tau\left\{\left(\frac{\partial \Phi}{\partial x}\right)_0+\left(\frac{\partial \Phi}{\partial y}\right)_0\cot\theta\right\}, \tag{A 3}$$

$$Y_1-Y_{-1}=-2\delta\tau\left\{\left(\frac{\partial \Phi}{\partial x}\right)_0\tan\theta+\left(\frac{\partial \Phi}{\partial y}\right)_0\right\}, \tag{A 4}$$

$$(\Phi_1-\Phi_{-1})_x=\delta\tau\left\{\left(\frac{\partial \Phi}{\partial x}\right)_0^2+\left(\frac{\partial \Phi}{\partial y}\right)_0^2+2y_0\right\}+\left(\frac{\partial \Phi}{\partial x}\right)_0(X_1-X_{-1}), \tag{A 5}$$

$$(\Phi_1-\Phi_{-1})_y=\delta\tau\left\{\left(\frac{\partial \Phi}{\partial x}\right)_0^2+\left(\frac{\partial \Phi}{\partial y}\right)_0^2+2y_0\right\}+\left(\frac{\partial \Phi}{\partial y}\right)_0(Y_1-Y_{-1}), \tag{A 6}$$

where θ is the slope of the boundary where it is intersected by the particular mesh line, and $(\partial\Phi/\partial x)_0$ and $(\partial\Phi/\partial y)_0$ are the derivatives of Φ_0 at that point in the x and y directions respectively.

In practice, however, the use of small steps in time and space is not very satisfactory, the accumulation of errors in a large number of time steps becoming very serious. The difficulties are particularly acute near the base, where the fluid spreads out rapidly. The determination of its boundary is then rather difficult, since, for any given mesh length δy in the vertical direction, a considerable horizontal range of the fluid may occupy a depth less than δy. If large time intervals are taken, on the other hand, the finite-difference approximations are not very sound.

At the time of this investigation, moreover, the assumption was made that the angle of contact between the fluid boundary and the base was 90°. The results are not therefore regarded as being any more satisfactory than those obtained by the method of § 2. One case was attempted, that of a plane column initially rectangular, of height one unit and breadth two units, in non-dimensional terms. Small time intervals were used for the first few steps, larger intervals later, and results were obtained up to a non-dimensional time of 1·5 units. A certain amount of smoothing was necessary, and it was estimated that the results would become almost meaningless, due to the accumulation of errors already mentioned, for subsequent times.

Figure 16 shows the calculated fluid boundaries at several instants of time in this range. In spite of the generally unsatisfactory nature of this work, there is some measure of agreement with experimental results.

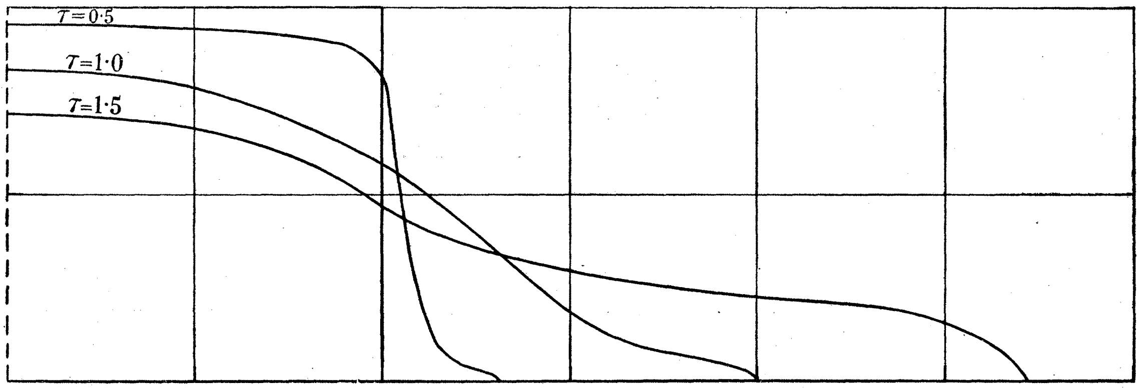

FIGURE 16. Collapse of a plane symmetric column *in vacuo*, initially rectangular at rest, at various specific times τ. The solution obtained by the Mathematics Division of the National Physical Laboratory, using relaxation methods.

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